

Symmetry Conditions on Nonlinear Spinor Field Functionals

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The symmetry conditions of nonlinear spinor theory of elementary particles, i.e. the definitions of quantum numbers, are given for the nonlinear spinor field functionals.

The operator equations of quantum theory can be replaced formally by functional equations of corresponding Schwinger functionals¹⁻³. To give this formalism a physical and mathematical meaning one has to develop a complete functional quantum theory as has been proposed in a preceding paper⁴. Then the complete physical information has to be given by functional operations only. Especially the quantum numbers of ordinary quantum theory have to be reproduced functionally. As the quantum numbers are defined by the eigenvalues of the generators of the corresponding invariance groups, one has to investigate these quantities in functional space. This is done in this paper. To have a definite model we consider the nonlinear spinor field with noncanonical relativistic Heisenberg quantization⁵ the form invariance group of which is the Poincaré group. Although this model has still other symmetry properties we restrict ourselves to the discussion of the quantum number conditions resulting from this group, as the considerations for other groups and models are quite analogous.

1. Fundamentals

First we discuss the basic group theoretical assumptions of nonlinear spinor theory as far as they are concerned with the Poincaré-group. For the other dynamical assumptions we refer to the literature^{5,6}.

a) In ordinary quantum theory with canonical quantization it is proven that the form invariance of the dynamical equations against certain transformation groups causes the corresponding Hilbert space to be a representation space of these groups. For noncanonical quantized theories this cannot be proven in general but has to be postulated. The Poincaré-group is a symmetry group in nonlinear spinor theory, therefore we postulate a Hilbert-space of physical states being a representation space of this group, e.g. we assume for any inhomogenous Lorentz transformation

$$x'_\mu = A^\nu_\mu x_\nu + a_\mu \quad (1.1)$$

a representation U in the corresponding Hilbert space of states with

$$|a\rangle' = U(A, a) |a\rangle \quad \text{and} \quad |0\rangle = U |0\rangle \quad (1.2), (1.2a)$$

for the physical ground state $|0\rangle$.

b) Under an infinitesimal transformation, the state vector $|a\rangle$ undergoes the transformation

$$|a\rangle' = \left(1 + i \varepsilon^k P_k + \frac{i}{2} \omega^{kl} M_{kl} \right) |a\rangle \quad (1.3)$$

where P_k , M_{kl} are the infinitesimal generators of the transformations $U(A, a)$.

¹ J. SCHWINGER, Proc. Nat. Acad. Sci. **37**, 452 [1951].

² D. LURIE, Particles and fields, Interscience Publ., New York 1968.

³ J. RZEWUSKI, Field theory, Vol. 2, Iliffe Books LTD, London 1969.

⁴ H. STUMPF, Z. Naturforsch. **24a**, 188 [1969].

⁵ W. HEISENBERG, An Introduction to the Unified Theory of Elementary Particles, Wiley and Sons, London 1967.

⁶ H. RAMPACHER, H. STUMPF u. F. WAGNER, Fortschr. Phys. **13**, 385 [1965].

⁷ S. GASIOROWICZ, Elementary particle physics, Wiley and Sons, London 1966.



They satisfy the commutation relations⁷

$$\begin{aligned} [\mathbf{M}_{kl}, \mathbf{P}_h]_- &= i(g_{lk}\mathbf{P}_h - g_{kh}\mathbf{P}_l), \\ [\mathbf{M}_{kl}, \mathbf{M}_{mn}]_- &= -i(g_{km}\mathbf{M}_{ln} - g_{lm}\mathbf{M}_{kn} + g_{kn}\mathbf{M}_{ml} - g_{ln}\mathbf{M}_{mk}), [\mathbf{P}_k, \mathbf{P}_l]_- = 0 \end{aligned} \quad (1.4)$$

with $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$, $g_{\mu\nu} = 0$, $\nu \neq \mu$. The irreducible representations of any group are characterized by the values of the invariant operators⁸. For the Poincaré group these invariants are

$$P^2 = P_\mu P^\mu, \quad W = \Gamma_\mu \Gamma^\mu \quad (1.5)$$

with

$$\Gamma_\mu = \frac{1}{2m} \varepsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}$$

i. e. the irreducible representations of the Poincaré group are characterized by the mass- and the spinvalues.

c) For the definition of quantum numbers one choses a complete set of commuting operators, which can be simultaneously diagonalized. It is convenient to choose the set $\{P^2, W, P_\mu, \Gamma_3\}$. (Mass m , total spin s , linear momentum p , spin direction s_3).

Taking for $|a\rangle$ an eigenstate of these operators, we get:

$$\mathbf{P}_\mu |a\rangle = p_\mu |a\rangle, \quad \mathbf{P}^2 |a\rangle = m^2 |a\rangle, \quad \mathbf{\Gamma}_\mu \mathbf{\Gamma}^\mu |a\rangle = s(s+1) |a\rangle, \quad \mathbf{\Gamma}_3 |a\rangle = s_3 |a\rangle. \quad (1.6a-d)$$

d) According to the dynamical assumptions of nonlinear spinor theory the states $|a\rangle$ have to be characterized by their projections on a set of base states constructed by the field operators of the theory. This gives the set of functions^{5,6}.

$$\tau(x_1 \dots x_n) := \langle 0 | T \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) | a \rangle \quad (1.7)$$

where $\psi_\alpha(x)$ is assumed to be a Hermitean field operator with the transformation property^{9,10}

$$U^{-1} \psi_\alpha(x') U = L^\beta_\alpha \psi_\beta(\Lambda^{-1}(x' - a)), \quad U^{-1} \psi^\alpha(x') U = L^{-1\alpha}_\beta \psi^\beta(\Lambda^{-1}(x' - a)). \quad (1.8)$$

Observing that all finite transformations can be generated by infinitesimal ones and using the results of App. I and App. II, we get the transformation properties of the functions (1.7):

$$\tau'(x'_1 \dots x'_n) = L^{\beta_1}_{\alpha_1} \dots L^{\beta_n}_{\alpha_n} \tau(\Lambda^{-1}(x_1 - a) \dots \Lambda^{-1}(x_n - a)). \quad (1.9)$$

where the transformed function τ' is defined with respect to $|a\rangle'$ of (1.2).

2. Symmetry Conditions in Configuration Space

According to d) we have to formulate the symmetry-conditions (1.6a-d) for the projections (1.7).

Chosing $|a\rangle$ as an element of the canonical basis as in c), the following relations hold:

$$\left[\sum_{j=1}^n P_\mu(x_j) - p_\mu \right] \tau_n(x_1 \dots x_n) = 0, \quad (2.1a)$$

$$\left[\left(\sum_{j=1}^n P_\mu(x_j) \right) \left(\sum_{j=1}^n P^\mu(x_j) \right) - m^2 \right] \tau_n(x_1 \dots x_n) = 0, \quad (2.1b)$$

$$\left[\frac{1}{4m^2} \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\mu\nu'\sigma'\rho'} \sum_i P^{\nu'}(x_i) \sum_j M^{\rho\sigma}(x_j) \sum_k P_{\nu'}(x_k) \sum_l M_{\rho'\sigma'}(x_l) - s(s+1) \right] \tau_n(x_1 \dots x_n) = 0, \quad (2.1c)$$

$$\left[\frac{1}{2m} \varepsilon_{3\nu\rho\sigma} \sum_i P^{\nu}(x_i) \sum_j M^{\rho\sigma}(x_j) - s_3 \right] \tau_n(x_1 \dots x_n) = 0 \quad (2.1d)$$

with $P_\mu(x)$, $M_{\rho\sigma}(x)$ acting in ordinary spinor space.

⁸ YU. M. SHIROKOV, Soviet Physics-JETP (New York) **6**, 33, 919, 929 [1958].

⁹ H. STUMPF, Z. Naturforsch. **25a**, 575 [1970].

¹⁰ H. P. DÜRR and F. WAGNER, Nuovo Cim. **46**, 223 [1966].

Proof. From (1.8) we get, in the special case of pure translations for infinitesimal transformations, $U(a) = 1 + i P^\mu a_\mu$

$$[P_\mu \psi(x)]_- = -i \partial_\mu \psi(x) = P_\mu(x) \psi(x). \quad (2.2)$$

For infinitesimal 4-rotations $U(A) = 1 + (i/2) a^{\mu\nu} M_{\mu\nu}$, $L(A) = 1 - (i/2) a^{\mu\nu} \Sigma_{\mu\nu}$ with $\Sigma_{\mu\nu} = - (1/4 i) \cdot [\gamma_\mu \gamma_\nu]$, we get from (1.8)

$$[M_{\mu\nu}, \psi(x)] - [\Sigma_{\mu\nu} + 1/i (-x_\mu \partial_\nu + x_\nu \partial_\mu)] \psi(x) = M_{\mu\nu}(x) \psi(x). \quad (2.3)$$

From (1.6a) we get

$$p_\mu \langle 0 | T \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) | a \rangle = \langle 0 | T \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}^\dagger(x_n) P_\mu | a \rangle. \quad (2.4)$$

Applying (2.2) and taking into account that $\sum_i P_\mu(x^i)$ commutes with the time ordering operator T (see App. II) yields:

$$\sum_i P_\mu(x_i) \langle 0 | T \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) | a \rangle = \langle 0 | T \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) P_\mu | a \rangle. \quad (2.5)$$

Combination of (2.4) and (2.5) gives (2.1a).

Applying this procedure once more yields (2.1b).

To prove (2.1c) we observe

$$\begin{aligned} s(s+1) \tau_n(x_1 \dots x_n) &= \langle 0 | T \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) \Gamma_\mu \Gamma^\mu | a \rangle \\ &= \langle 0 | T \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) \frac{1}{4m^2} \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\mu\nu\rho'\sigma'} P^\mu M^{\rho\sigma} P_{\nu'} M_{\rho'\sigma'} | a \rangle \end{aligned} \quad (2.6)$$

$\sum_i M_{\mu\nu}(x_i)$ commutes with the time-ordering operator T (see App. II). Then the same procedure leading to (2.5) yields:

$$s(s+1) \tau_n(x_1 \dots x_n) = \frac{1}{4m^2} \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\mu\nu\rho'\sigma'} \sum_i P^\nu(x^i) \sum_j M^{\rho\sigma}(x_j) \sum_k P_{\nu'}(x_k) \sum_l M_{\rho'\sigma'}(x_l) \tau_n(x_1 \dots x_n) \quad (2.7)$$

i.e. (2.1c).

Applying the same considerations to Γ_3 yields (2.1d).

3. Symmetry Conditions in Functional Space

It is convenient and for a thorough investigation necessary to describe the set of functions (1.11) by functionals. They are defined by

$$\mathfrak{T}(j) |0\rangle_F := \sum_{n=0}^{\infty} \frac{j^n}{n!} \int \tau_n(x_1 \dots x_n) j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n) dx_1 \dots dx_n |0\rangle_F =: |\mathfrak{T}(j)\rangle \quad (3.1)$$

where $|0\rangle_F$ is the functional groundstate⁹.

For the sources the following commutation relations are assumed⁹

$$[j_\alpha(x), j_{\alpha'}(x')]_+ = [\partial_\alpha(x), \partial_{\alpha'}(x')]_+ = 0, \quad [j_\alpha(x), \partial_{\alpha'}(x')]_+ = \delta_\alpha^{\alpha'} \delta(x - x'). \quad (3.2)$$

Additionally we require:

$$\partial_\alpha(x) |0\rangle_F = 0. \quad (3.3)$$

The transformation properties of the sources are assumed to be⁹

$$V^{-1} j_\alpha(x') V = L_\alpha^\beta j_\beta(A^{-1}(x' - a)), \quad V^{-1} j^\alpha(x') V = L^{-1\alpha}_\beta j^\beta(A^{-1}(x' - a)). \quad (3.4)$$

Together with (1.9) and the invariance of $|0\rangle_F$ under transformations in functional space this causes for the field functional (3.1) the transformation property:

$$\mathfrak{T}'(j) |0\rangle_F = V \mathfrak{T}(j) |0\rangle_F \quad (3.5)$$

i.e. $\mathfrak{T}(j) |0\rangle_F$ transforms as a state in functional space.

To transfer the symmetry conditions (2.1a–d) in functional space we observe under the assumption (3.3) the following relation to hold

$$\int j_\alpha(x) g(x) \partial^\alpha(x) |\mathfrak{T}(j)\rangle = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \sum_{j=1}^n g(x_j) \tau_n(x_1 \dots x_n) j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n) dx_1 \dots dx_n |0\rangle_F. \quad (3.6)$$

Defining now the operators

$$\mathfrak{P}_k := \int j_\alpha(x) P_k(x) \partial^\alpha(x) dx, \quad \mathfrak{M}_{kl} := \int j_\alpha(x) M_{kl}(x) \partial^\alpha(x) dx \quad (3.8)$$

and

$$\mathfrak{G}^\mu := \frac{1}{2m} \varepsilon^{\mu\nu\rho\sigma} \mathfrak{P}_\nu \mathfrak{M}_{\rho\sigma} \quad (3.9)$$

where m is the eigenvalue of \mathbf{P}^2 it can be shown easily that the symmetry conditions (2.7) to (2.10) go over into the conditions

$$\mathfrak{P}_k |\mathfrak{T}(j)\rangle = p_k |\mathfrak{T}(j)\rangle, \quad \mathfrak{P}^2 |\mathfrak{T}(j)\rangle = m^2 |\mathfrak{T}(j)\rangle, \quad (3.10), (3.11)$$

$$\mathfrak{G}_\mu \mathfrak{G}^\mu |\mathfrak{T}(j)\rangle = s(s+1) |\mathfrak{T}(j)\rangle, \quad \mathfrak{G}_3 |\mathfrak{T}(j)\rangle = s_3 |\mathfrak{T}(j)\rangle, \quad (3.12), (3.13)$$

(3.10) to (3.13) are the general symmetry conditions in functional space. For any calculational process of functional eigenstates in nonlinear spinor theory these symmetry conditions have to be satisfied in order to define a complete set of quantum numbers. For practical calculations however the conditions (3.10) to (3.13) can be simplified considerably if one uses the rest frame of the entire system. To evaluate this in detail we observe the relations (1.4) to be valid in any representation of the Lorentz group. By this it can be easily proven, using (2.5) and the corresponding relation for \mathbf{M}_{kl} together with (3.6), that

$$[\mathfrak{M}_{kl}, \mathfrak{P}_h]_- |\mathfrak{T}(j)\rangle = i(g_{lh} \mathfrak{P}_k - g_{kh} \mathfrak{P}_l) |\mathfrak{T}(j)\rangle, \quad (3.14)$$

$$[\mathfrak{P}_h, \mathfrak{P}_j]_- |\mathfrak{T}(j)\rangle = 0 \quad (3.15)$$

is valid, too. These relations can be used to simplify the conditions (3.10) to (3.13) in the rest frame. There we have

$$\mathfrak{P}_0 |\mathfrak{T}(j)\rangle = m |\mathfrak{T}(j)\rangle, \quad \mathfrak{P}_i |\mathfrak{T}(j)\rangle = 0 \quad (3.16), (3.17)$$

and by applying (3.14), (3.15), (3.16), (3.17) equation (3.12) goes over into

$$\mathfrak{G}_\mu \mathfrak{G}^\mu |\mathfrak{T}(j)\rangle = [\mathfrak{M}_{23}^2 + \mathfrak{M}_{31}^2 + \mathfrak{M}_{12}^2] |\mathfrak{T}(j)\rangle = s(s+1) |\mathfrak{T}(j)\rangle \quad (3.18)$$

while (3.13) becomes

$$\mathfrak{M}_{12} |\mathfrak{T}(j)\rangle = s_3 |\mathfrak{T}(j)\rangle. \quad (3.19)$$

Now the symmetry conditions (3.16), (3.17), (3.18), (3.19) are very simple ones. They have been reduced to a nonrelativistic problem, which can be solved by standard methods.

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Appendix I

In this appendix we discuss the possibility to assume a transformation property like (1.9) for classical spinor functions. This question is not trivial in so far, as classical spinor functions are known only for free fields, whereas no classical solution for any nonlinear equation has been found. To proceed we assume a classical spinor function $f_\alpha(x)$ which transforms according to

$$f'_\alpha(x') = L^{-1\beta}_\alpha(A) f_\beta(Ax + a). \quad (I.1)$$

To show the selfconsistence of (I.1) we perform a Fourier-Transformation

$$f_\alpha(x) = \int e^{-ipx} \tilde{f}_\alpha(p) dp. \quad (I.2)$$

This integral over the total Lorentzspace can be lorentzinvariant decomposed in to

$$f_\alpha(x) = \int_{L_z} e^{-ipx} \tilde{f}_\alpha(p) dp + \int_{L_R} e^{-ipx} \tilde{f}_\alpha(p) dp \quad (\text{I.3})$$

where L_z allows only p -vectors with $-p^2 + p_0 = m^2 \geq 0$, whereas L_R allows only space like vectors p with $-p^2 + p_0^2 = -m^2 < 0$. Then the integrals in (I.3) can be decomposed further to give

$$\int_{L_z} e^{-ipx} \tilde{f}_\alpha(p) dp = \int_0^\infty dm \int_{-\infty}^\infty dp \exp\{\mp i p r \pm i p_0 t\} \tilde{f}_\alpha(p, m) \frac{m}{(m^2 + p^2)^{1/2}}, \quad (\text{I.4})$$

and

$$\int_{L_R} e^{-ipx} \tilde{f}_\alpha(p) dp = \int_0^\infty dm \int_{-\infty}^\infty dp_0 \int d\Omega \exp\{\mp i p r \pm i p_0 t\} \tilde{f}_\alpha(p, p_0) \quad (\text{I.5})$$

Now a thorough study of the Dirac equation shows, that it can be solved equally well for imaginary masses and allows a set of fundamental solutions for different spin. The only difference to the case of real mass is the different transformation property of the scalar product which is not invariant against time reflections. Ignoring for the moment this drawback, which possibly could have a deeper meaning for the interaction, we are able to evaluate (I.4), (I.5) by means of eigensolutions of the Dirac equation.

$$\begin{aligned} \int_{L_z} e^{-ipx} \tilde{f}_\alpha(p) dp &= \int_0^\infty dm \int_{R_3} \exp\{i p r - i p_0 t\} \sum_{r=1}^2 b_r(p, m) w_{\alpha^r}(p, m) dp \\ &\quad + \int_0^\infty dm \int_{R_3} \exp\{-i p r + i p_0 t\} \sum_{n=1}^2 d_r(p, m) v_{\alpha^r}(p, m) dp \end{aligned} \quad (\text{I.6})$$

and

$$\begin{aligned} \int_{L_R} e^{-ipx} \tilde{f}_\alpha(p) dp &= \int_0^\infty dm \int_{-\infty}^\infty dp_0 \int d\Omega \exp\{i p r - i p_0 t\} \sum_{r=1}^2 h_r(p, p_0) u_{\alpha^r}(p, p_0) \\ &\quad + \int_0^\infty dm \int_{-\infty}^\infty dp_0 \int d\Omega \exp\{-i p r + i p_0 t\} \sum_{r=1}^2 k_r(p, p_0) z_{\alpha^r}(p, p_0) \end{aligned} \quad (\text{I.7})$$

as any component of the spectral decomposition transforms like (I.1). (I.1) itself is a selfconsistent assumption which does not lead to any contradiction.

Appendix II

In this appendix we prove the following statements:

- a) $\sum_i P_\mu(x_i) \tau_n(x_1 \dots x_n) = \langle 0 | T \sum_i P_\mu(x_i) \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) | a \rangle,$
- b) $\sum_i M_{\mu\nu}(x_i) \tau_n(x_1 \dots x_n) = \langle 0 | T \sum_i M_{\mu\nu}(x_i) \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) | a \rangle.$

For the proof of a) it is sufficient to consider only $\sum_i P_0(x_i) = -i \sum_i \partial_0(x_i)$ because the other components are not affected by the time ordering. We have:

$$\begin{aligned} &\sum_i \partial_0(x_i) p \sum_{\lambda_1 \dots \lambda_n} (-1)^p \Theta(t_{\lambda_1} - t_{\lambda_2}) \dots \Theta(t_{\lambda_{n-1}} - t_{\lambda_n}) \langle 0 | \psi_{\alpha_{\lambda_1}}(x_{\lambda_1}) \dots \psi_{\alpha_{\lambda_n}}(x_{\lambda_n}) | a \rangle \\ &= p \sum_{\lambda_1 \dots \lambda_n} (-1)^p \langle 0 | \psi_{\alpha_{\lambda_1}}(x_{\lambda_1}) \dots \psi_{\alpha_{\lambda_n}}(x_{\lambda_n}) | a \rangle \sum_i \partial_0(x_i) \Theta(t_{\lambda_1} - t_{\lambda_2}) \dots \Theta(t_{\lambda_{n-1}} - t_{\lambda_n}) \\ &\quad + \langle 0 | T \sum_i \partial_0(x_i) \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) | a \rangle. \end{aligned} \quad (\text{II.1})$$

Obviously $\sum_i \partial_0(x_i) p \sum_{\lambda_1 \dots \lambda_n} (-1)^p \Theta(t_{\lambda_1} - t_{\lambda_2}) \dots \Theta(t_{\lambda_{n-1}} - t_{\lambda_n})$ vanishes for any permutation, therefore statement a) is valid.

The proof of b) reduces to the proof of

$$\sum x_\mu^i \partial_0(x_i) \langle 0 | T \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) | a \rangle = \langle 0 | T \sum x_\mu^i \partial_0(x_i) \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n) | a \rangle \quad (\text{II.2})$$

because the other parts of $\sum_i M_{\mu\nu}(x_i)$ are not affected by the time ordering. To prove (II.2) we start with the relation

$$(\partial/\partial t) T x_1 \dots x_n x = T x_1 \dots \dot{x}_n x + \sum_{r=1}^n (-1)^r \delta(x - x_r) T x_1 \dots x_{r-1} x_{r+1} \dots x_n \quad (\text{II.3})$$

(see 9, (III.8), where we have used the abbreviation

$$T x_1 \dots x_n := T \psi_{\alpha_1}(x_1) \dots \psi_{\alpha_n}(x_n).$$

From the antisymmetry of the T -product and (II.3) follows:

$$\begin{aligned} \sum_{i=1}^n x_\mu^i \partial_0(x_i) T x_1 \dots x_n &= \sum_{i=1}^n (-1)^{n-i} x_\mu^i \partial_0(x_i) T x_1 \dots x_{i-1} x_{i+1} \dots x_n x_i \\ &= T \sum_{i=1}^n x_\mu^i \partial_0(x_i) x_1 \dots x_n \\ &\quad + \sum_{i=1}^n (-1)^{n-i} x_\mu^i \left[\sum_{r=1}^{i-2} (-1)^r \delta(x_i - x_r) T x_1 \dots x_{r-1} x_{r+1} \dots x_{i-1} x_{i+1} \dots x_n \right. \\ &\quad + (-1)^{i-1} \delta(x_i - x_{i-1}) T x_1 \dots x_{i-2} x_{i+1} \dots x_n + (-1)^i \delta(x_i - x_{i+1}) T x_1 \dots x_{i-1} x_{i+2} \dots x_n \\ &\quad \left. + \sum_{r=i+2}^n (-1)^{r-1} \delta(x_i - x_r) T x_1 \dots x_{i-1} x_{i+1} \dots x_{r-1} x_{r+1} \dots x_n \right]. \end{aligned} \quad (\text{II.4})$$

Now we observe

$$\begin{aligned} &\sum_{i=1}^n (-1)^{n+1} x_\mu^i \delta(x_i - x_{i-1}) T x_1 \dots x_{i-2} x_{i+1} \dots x_n + \sum_{i=1}^n (-1)^n x_\mu^i \delta(x_i - x_{i+1}) T x_1 \dots x_{i-1} x_{i+2} \dots x_n \quad (\text{II.5}) \\ &= \sum_{i=2}^n (-1)^{n+1} x_\mu^i \delta(x_i - x_{i-1}) T x_1 \dots x_{i-2} x_{i+1} \dots x_n + \sum_{i=1}^{n-1} (-1)^n x_\mu^i \delta(x_i - x_{i+1}) T x_1 \dots x_{i-1} x_{i+2} \dots x_n \\ &= \sum_{i=1}^{n-1} (-1)^{n+1} x_\mu^{i+1} \delta(x_{i+1} - x_i) T x_1 \dots x_{i-1} x_{i+2} \dots x_n + \sum_{i=1}^{n-1} (-1)^n x_\mu^i \delta(x_i - x_{i+1}) T x_1 \dots x_{i-1} x_{i+2} \dots x_n = 0 \end{aligned}$$

and

$$\begin{aligned} &\sum_{i=1}^n (-1)^{n-i} x_\mu^i \left[\sum_{r=1}^{i-2} (-1)^r \delta(x_i - x_r) T x_1 \dots x_{r-1} x_{r+1} \dots x_{i-1} x_{i+1} \dots x_n \right. \\ &\quad \left. + \sum_{r=i+2}^n (-1)^{r+1} \delta(x_i - x_r) T x_1 \dots x_{i-1} x_{i+1} \dots x_{r-1} x_{r+1} \dots x_n \right] \\ &= \sum_{i=3}^n (-1)^{n-i} x_\mu^i \sum_{r=1}^{i-2} (-1)^r \delta(x_i - x_r) T x_1 \dots x_{r-1} x_{r+1} \dots x_{i-1} x_{i+1} \dots x_n \\ &\quad + \sum_{i=1}^{n-2} \sum_{r=i+2}^n (-1)^{n+r-i-1} x_\mu^i \delta(x_i - x_r) T x_1 \dots x_{i-1} x_{i+1} \dots x_{r-1} x_{r+1} \dots x_n \\ &= \sum_{r=1}^{n-2} \sum_{i=r+2}^n (-1)^{n-i+r} x_\mu^i \delta(x_i - x_r) T x_1 \dots x_{i-1} x_{i+1} \dots x_{r-1} x_{r+1} \dots x_n + \text{second term} \quad (\text{II.6}) \\ &= \sum_{i=1}^{n-2} \sum_{r=i+2}^n (-1)^{n-r+i} x_\mu^i \delta(x_i - x_r) T x_1 \dots x_{i-1} x_{i+1} \dots x_{r-1} x_{r+1} \dots x_n + \text{second term} = 0. \end{aligned}$$

Therefore statement b) holds too.